THE FOKKER–PLANCK EQUATION IN THE SECOND ORDER PITCH ANGLE APPROXIMATION AND ITS EXACT SOLUTION

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In contrast to the Boltzman scattering of particles, when the existence of the unscattered particles is possible, the Fokker-Plank scattering can not be used to identify the unscattered particles on the background of the scattered ones. For the case of the Boltzman scattering of particles the kinetic equation describing this process may be solved exactly for all angles. From the other hand the Fokker-Plank kinetic equation is more complex. Here the exact analytical solution of the Fokker-Plank kinetic equation to the second order pitch-angle approximation is given.

Let us consider the following equation:

$$\partial_t f + v\cos\theta\ \partial_x f = \frac{v}{\lambda\sin\theta}\ \partial_\theta \sin\theta\ \partial_\theta f + \delta(x)\ \delta(t)\ \frac{\delta(\theta-\theta_0)}{\sin\theta_0}\,. \tag{1}$$

Here θ is the particle pitch angle (the angle between velocity vector and the axis x) , λ is the particle mean free path ($\lambda = v/v_s$, v_s is the scattering frequency). Introducing the dimensionless variables $y = x/\lambda$ and $\tau = vt/\lambda$, Eq. (1) can be written in the form

$$\partial_\tau f + \mu\ \partial_y f = \partial_\mu f(1-\mu^2)\ \partial_\mu f + \frac{1}{\lambda}\delta(y)\ \delta(\tau)\ \delta(\mu-\mu_0)\,, \tag{2}$$

where $\mu = \cos\theta$. Taking the Fourier transform of the space variable y and the Laplace transform of the time τ, Eq. (2) gives

$$\partial_\mu(1-\mu^2)\ \partial_\mu f - \mathrm{i}k\mu f - sf = -\frac{1}{\lambda}\delta(\mu-\mu_0)\,. \tag{3}$$

Solutions of Eq. (3) are expressed by the Coulomb – spheroidal functions with indefinite parameters [1] and some approximations are necessary. The simplest approximation corresponds to very small pitch angle θ, when one can put $\sin\theta \to \theta$ and $\cos\theta \to 1$. In the other words the equality $\sin^2\theta + \cos^2\theta = 1$ is fulfilled up only to the first order of θ. In this case one find

$$\partial_\tau f + \partial_y f = \frac{1}{\theta}\ \partial_\theta\ \theta\ \partial_\theta f + \frac{1}{\lambda}\delta(y)\ \delta(\tau)\frac{\delta(\theta-\theta_0)}{\theta_0}\,. \tag{4}$$

This equation including the focusing term has been investigated by Toptygin *et al.* [2] and Kats *et al.* [3, 4] for the stationary and non-stationary case, respectively. The solution of Eq. (4) reads

$$f(y,\tau,\theta,\theta_0) = \frac{1}{2\lambda\tau}\ \exp\left[-\frac{\theta^2+\theta_0^2}{4\tau}\right]\ I_0\left(\frac{\theta\theta_0}{2\tau}\right)\ \delta(y-\tau)\,, \tag{5}$$

where $I_0(x)$ is the zero order Bessel function of imaginary argument (the hyperbolic Bessel function). The last equation rewritten in the variables $\{x,t\}$ reads

$$f(x,t,\theta,\theta_0) = \frac{\lambda}{2vt}\ \exp\left[-\lambda\frac{\theta^2+\theta_0^2}{4vt}\right]\ I_0\left(\lambda\frac{\theta\theta_0}{2\tau}\right)\ \delta(x-vt)\,. \tag{6}$$

The δ-function in Eq. (6) expresses the particle free propagation along the axis x.

Let us consider the second order pitch angle approximation. It means that $\sin\theta \to \theta$ and $\cos\theta \to (1-\theta^2/2)$. The equality $\sin^2\theta + \cos^2\theta = 1$ is fulfilled up to the second order of θ and Eq. (1) reads

$$\partial_\tau f + \left(1 - \frac{\theta^2}{2}\right) \partial_y f = \frac{1}{\theta}\, \partial_\theta \theta\, \partial_\theta f + \frac{1}{\lambda}\, \delta(y)\, \delta(\tau)\, \frac{\delta(\theta-\theta_0)}{\theta_0}\,. \tag{7}$$

Substituting $\eta = \theta^2/4$ changes Eq. (6) to the following

$$\partial_\tau f + (1-2\eta)\, \partial_y f = \partial_\eta\, \eta\, \partial_\eta f + \frac{1}{2\lambda}\, \delta(y)\, \delta(\tau)\, \delta(\eta-\eta_0)\,. \tag{8}$$

Taking the Laplace transform of τ and Fourier transform of y one obtains the equation for $f(s,k,\eta,\eta_0)$:

$$\partial_\eta\, \eta\, \partial_\eta f + 2\mathrm{i}k\eta f - (s+\mathrm{i}k) f = -\frac{1}{2\lambda}\, \delta(\eta-\eta_0)\,. \tag{9}$$

Solution of Eq. (9) can be the generalized form of the previous solution (5). For this purpose the solution is sought in the form

$$f = \frac{1}{2\lambda}\int_0^\infty \exp[-\alpha(\eta+\eta_0)\cosh\xi] \left(\coth\frac{\xi}{2}\right)^\beta I_0(2\alpha\sqrt{\eta\eta_0}\sinh\xi)\mathrm{d}\xi \tag{10}$$

with unknown parameters α, β. These are determined the substituting (10) into Eq. (9):

$$\alpha = \sqrt{-2\mathrm{i}k}, \qquad\qquad \beta = -\frac{s+\mathrm{i}k}{\sqrt{-2\mathrm{i}k}}\,. \tag{11}$$

Conditions for convergence of the integral (10) are the following: Re $\alpha > 0$, Re $\beta < 0$. Then $\alpha = (1-\mathrm{i})\sqrt{k}$ for $k > 0$ and $\alpha = (1+\mathrm{i})\sqrt{k}$ for $k < 0$, respectively. The inverse Laplace transform exists under condition Re s > 0, therefore, the condition Re $\beta > 0$ is fulfilled for those α and one obtains

$$\begin{aligned} f(k,\tau,\eta,\eta_0) &= \frac{1}{2\lambda}\, \exp\left[-\mathrm{i}k\tau - \alpha(\eta+\eta_0)\, \coth(\alpha\tau)\right] \\ &\times\; I_0\left(\frac{2\alpha\sqrt{\eta\eta_0}}{\sinh(\alpha\tau)}\right) \frac{2\alpha}{\sinh(\alpha\tau)}\,. \end{aligned} \tag{12}$$

The function of the argument k in Eq. (12) is regular relative to the complex value of k, so, one can apply the inverse Fourier transform of (12). Then the solution of Eq. (8) is

$$\begin{aligned} f(y,\tau,\eta,\eta_0) &= \frac{1}{4\pi\lambda}\int_0^\infty \Big\{\exp\Big[\mathrm{i}ky - \mathrm{i}k\tau - (1+\mathrm{i})\sqrt{k}(\eta+\eta_0) \\ &\times\; \coth\Big((1+\mathrm{i})\sqrt{k}\tau\Big)\Big] \\ &\times\; I_0\left(\frac{2(1+\mathrm{i})\sqrt{k}\sqrt{\eta\eta_0}}{\sinh\left[(1+\mathrm{i})\sqrt{k}\tau\right]}\right) \frac{2(1+\mathrm{i})\sqrt{k}}{\sinh\left[(1+\mathrm{i})\sqrt{k}\tau\right]} \\ &+\; \exp\Big[-\mathrm{i}ky + \mathrm{i}k\tau - (1-\mathrm{i})\sqrt{k}(\eta+\eta_0)\coth\Big((1-\mathrm{i})\sqrt{k}\tau\Big)\Big] \\ &\times\; I_0\left(\frac{2(1-\mathrm{i})\sqrt{k}\sqrt{\eta\eta_0}}{\sinh\left[(1-\mathrm{i})\sqrt{k}\tau\right]}\right) \frac{2(1-\mathrm{i})\sqrt{k}}{\sinh\left[(1-\mathrm{i})\sqrt{k}\tau\right]}\Big\}\, dk\,, \end{aligned} \tag{13}$$

or in the $\{x,t,\theta,\theta_0\}$ variables,

$$
\begin{aligned}
f(x,t,\theta,\theta_0) &= \frac{1}{4\pi\lambda}\int_0^\infty \left\{\exp\left[\frac{\mathrm{i}k}{\lambda}(x-vt)-(1+\mathrm{i})\frac{\sqrt{k}}{4}(\theta^2+\theta_0^2)\right.\right.\\
&\times \left.\coth\left((1+\mathrm{i})\sqrt{k}\frac{vt}{\lambda}\right)\right]\\
&\times I_0\left(\frac{(1+\mathrm{i})\sqrt{k}\theta\theta_0}{2\sinh\left[(1+\mathrm{i})\sqrt{k}vt/\lambda\right]}\right)\frac{2(1+\mathrm{i})\sqrt{k}}{\sinh\left[(1+\mathrm{i})\sqrt{k}vt/\lambda\right]}\\
&+ \exp\left[-\mathrm{i}\frac{k}{\lambda}(x-vt)-(1-\mathrm{i})\frac{\sqrt{k}}{4}(\theta^2+\theta_0^2)\coth\left((1-\mathrm{i})\sqrt{k}\frac{vt}{\lambda}\right)\right]\\
&\times \left.I_0\left(\frac{(1-\mathrm{i})\sqrt{k}\theta\theta_0}{2\sinh\left[(1-\mathrm{i})\sqrt{k}vt/\lambda\right]}\right)\frac{2(1-\mathrm{i})\sqrt{k}}{\sinh\left[(1-\mathrm{i})\sqrt{k}vt/\lambda\right]}\right\}dk\,. \qquad (14)
\end{aligned}
$$

Eq. (6) can be obtained from (14) as a limit of the small time t. In fact, if t$\to 0$ Eq. (14) acquires the form

$$
f(x,t,\theta,\theta_0)=\frac{\lambda}{2vt}\exp\left[-\frac{(\theta^2+\theta_0^2)\lambda}{4vt}\right]I_0\left(\frac{\theta\theta_0\lambda}{2vt}\right)\delta\left(x-vt\left(1-\frac{\theta^2+\theta_0^2}{6}\right)\right)\,. \qquad (15)
$$

One can see from Eq. (15) that both processes of the pitch – angle diffusion and particle transport are connected with each other. To separate them the expansion of cosine into series up to the second order, $\cos\theta=1-\theta^2/2$, must be used and after that the term of θ^2 can be neglected in comparison with unit. This gives the Eq. (6).

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